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THE BAKSHĀLĪ MANUSCRIPT

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§ 1. Introductory

In spite of the publication by the Archæological Survey of India, of the text of the Bakshālī Manuscript † with elaborate introduction and notes by G. R. Kaye, there are perhaps many fresh points tending to throw additional light on the contents and the age of the manuscript. A considerable portion of the analysis of the manuscript was really due to Hoernle, and by an irony of fate, the work was left for completion to one whose views and prejudices were poles apart from those of the pioneer. It is regretted that Kaye's examination apparently scientific, is warped by his profound obsession that he should find traces of Greek influence in everything Indian. This attitude has vitiated most of his conclusions and made it worth while for others * to re-examine his statements and give them their proper weight. The present paper is an attempt at such re-examination and re-assessment.

It is now sufficiently known that a manuscript of 70 leaves of birch-bark, each 7″ by 4″, dealing mainly with Arithmetic was discovered in 1881 by a peasant, near a village called Bakshali about 70 miles from the famous Taxila in the north-western frontier of India. This manuscript has since been christened "The Bakshālī Manuscript." The text is written in Sarada script the date of which is highly disputed and the language is the so-called *Gatha* dialect a form of north-western *Prakrit*. Apart from the script and the language, there are many surprises in the mathematical contents

† The Bakshālī Manuscript by G. R. Kaye, *Archæological Survey of India*, New Imperial Series, Vol. XLIII, Parts I and II (1927) and Part III (1933). Hereafter referred to as "B. M. I, II, III."

* [illegible] *The Bakshālī Mathematics*, by Bibhutibhusan Datta, 1929; hereafter referred to as "Datta."

which make one hesitate in fixing its age. These surprises and peculiarities will now be studied under seven heads:

(i) The method of presentation,

(ii) Peculiar terminology, abbreviations, and the cross-symbol for subtraction,

(iii) The Decimal Notation and the absence of word-numerals,

(iv) The symbol for the unknown,

(v) The Rule of *Regula Falsi*,

(vi) The square-root rule and the process of reconciliation,

and (vii) Series and sequences.

§ 2. The Method of Presentation

Rules and examples are presented in verse in Sloka metre and the explanations are given in prose according to a certain stereotyped convention as follows:

(1) The re-statement of the problem in a symbolic notation—called Sthāpana or Nyāsa.

(2) The working proper—called Karaṇa

and (3) Verification—called Pratyaya.

This bears some analogy to the Euclidean scheme of

(1) particular enunciation corresponding to Nyāsa,

(2) construction corresponding to Karaṇa,

and (3) demonstration corresponding to verification.

It is interesting to note that among the early arithmeticians verification had played an important part and served almost as an equivalent of 'logical proof'. The majority of minds are content with the working rule, provided verification shows them that the results are correct. They do not care to worry themselves about the rationale. In actual teaching again, we find a pupil more easily convinced by verification than by a closely reasoned mathematical argument too difficult to follow. Thus, verification is a potent instrument in the pedagogy of mathematics. No wonder, therefore, that a mathematics teacher of bygone days should have thought of incorporating 'verification' as an act of faith in a work intended especially for the benefit of young pupils†. In the Bakshālī Manuscript we have perhaps a glimpse of a sort of teaching notes—something intermediary to an original treatise and a regular commentary—of a private tutor. The loose colloquial style adopted is also in keeping

† Vide the Colophon. B. M. II 50, recto. P. 142.

with this idea. One may also recollect in this connection that Bakshālī was near the famous ancient University centre Taxila.

'Verification' was common enough among the Indian mathematical commentators as early as the time of Chaturveda Prithūdakaswami of the 9th century A. D. The rule of Vyastavidhi or reversing the steps, discussed in almost all ancient Indian mathematical works is perhaps intended to be useful in 'verification', especially of the roots of equations and in the so-called 'think of a number' problems. For example,* we have the result $1+\frac{1}{2}\{2+\frac{1}{2}(3+\frac{3}{2}\cdot\overline{4+\frac{3}{2}\cdot\overline{5+\frac{4}{2}\cdot\frac{1}{4}}})\}=\frac{93}{16}$ verified by the principle of reversal of steps, viz.

$$\frac{2}{4}\left[\frac{2}{3}\left[\frac{2}{2}\left[2\left\{2\left(\frac{93}{16}-1\right)-2\right\}-3\right]-4\right]-5\right]=\frac{1}{4},$$ worked up from the innermost bracket outwards step by step.

§ 3. Peculiar Terminology etc.

The mathematical terminology adopted in the Bakshālī Manuscript though generally the same as in other Hindu Mathematical works, contains a few exceptions incidental more or less to the language or dialect employed. For example, the terms such as 'Partha', 'Dhanta', 'Pasta' are peculiar to the Gatha dialect. The use of the terms 'Savarna', 'Kalāsavarna', and 'Sadris'īkarana' with reference to fractions remind one of similar uses in other Hindu works.

There are several abbreviations, some of which are consistently employed, while others are used without any uniformity. For example, the abbreviation *yu* for *yutam* (for addition) is sometimes put in between and sometimes after the expressions to be summed:

** e.g.

ᶜ	5	*yu*ᶜ	and	11	*yu*ᶜ	5
1	1			1		1

meaning respectively $x+5$ and $11+5$.

The abbreviation *gu* for guṇitam (multiplication) is sometimes put in but very often dropped:

*** e.g.

2	5	3	*gu*ᶜ	7	4	*gu* · · · · · ·
1	2÷	1		2÷	1	

The text is mutilated here but it is clear that *gu* is omitted between $\frac{2}{1}$ and $\frac{5}{2}$÷.

* B. M. III pp. 187, 188; 67 verso and recto.

** B. M. III Folio 59 recto; p. 215.

*** B. M. III Folio 25 verso; p. 193; last line above the foot-notes.

It is likely that many abbreviations are 'ad-hoc' inventions of the scribes themselves, who must belong to a period much later than the original author and one need not therefore attach to them any historical or mathematical importance.

There is one unique symbol whose use is fairly consistent but for stray exceptions evidently due to the carelessness of the scribes. The sign of the cross (†) is used to denote Riṇa or negative and placed to the right of the number which it qualifies. The symbol is peculiar to the Bakshālī Manuscript and does not occur elsewhere in Hindu mathematical works. It is also an operational symbol for subtraction. In the Nyasa for a problem on profit and loss,* we have (†) after 56 in

16	6	riṇam 56†
5	1	1

to indicate that 56 is the loss. Again in another, ** we have

1	2	3	6
2	5	·	4†

where the significance of the symbol † is not clear. Apart from these two anomalies, wherever the cross † is used, the number preceding it is meant to be subtracted. On account of some resemblance (suggested first by Thibaut) between this cross-mark and the Diophantine mark ⋔ used for a similar purpose but in a different position, Kaye is inclined to perceive here a Greek influence; while Hoernle suggests that the Indian symbol may be an abbreviation of one of the numerous words riṇa, kanita, ūna, kshaya meaning diminution and makes a particular reference to *ka*, written exactly like the Bakshālī cross-symbol in the Asoka script. But Kaye gravely points out the danger of tracing an isolated symbol back through the ages, while he himself complacently falls into the same error with respect to the Diophantine symbol. In this connection we may recollect D. E. Smith's[4] warning that the Arithmetic of Diophantus now extant was written in the 13th century (a thousand years after the original) and we have to allow for the possible interpolations of medieval copyists. In the early medieval period in India, say from the time of Bhaṣkara, the dot-symbol (·) for the negative became the fashion. In a subsequent section we discuss two interesting uses of the dot-symbol in the Bakshālī Manuscript. The absence of the use of the dot-symbol for negative quantities is an important evidence that the Bakshālī work must have preceded Bhaskara.

* B. M. III p. 220. 63 recto.   ** B. M. III 18 recto. p. 211.

§ 4. The Decimal Notation and the absence of Word-numerals

Throughout the Bakshālī Manuscript, some special numerals are used and the decimal notation employed in much the same way as in the modern notation. There is no attempt made to mention big numbers in words. The only other early Indian work systematically avoiding word-numerals is the[5] Aryabhatīya containing an ad-hoc notation for expressing big numbers. The fashion of word-numeration, i.e., using a word to connote the idea of a number, say 'eyes' for '2', 'tithi' (lunar days of the half month) for '15', 'teeth' for '32' and so on, was set by Varāhamihira in the sixth century and since then became an extraordinarily popular method of numeration in Hindu mathematical and other works. The systematic elimination of word-numerals in the present work is a sure index that it belongs to a period when the word-numeration had not yet become the general fashion. The employment of decimal notation in the present work is an additional evidence that the notation flourished in India even in the early centuries of the Christian era. But Kaye argues the other way about and ascribes the work to the medieval period on account of the decimal notation. Every Hindu work from the earliest times including the Vedic period shows intimate familiarity with the decimal numeration, one, ten, hundred, thousand, etc. which naturally led sooner or later unmistakably to the decimal notation with its ten figures and place-value. The play of position is a conspicuous feature of early Indian arithmetical symbolism.[2]

§ 5. The Symbol for the Unknown

In the Bakshālī Manuscript the dot (·) is used with apparently two different kinds of significance but with the same underlying idea of 'void', 'gap', or 'emptiness'. The dot, primarily a symbol of 'emptiness' must have become secondarily a symbol for the unknown or absent quantity. For, in one of the Sūtras* we are directed to put any desired number in the place of the unknown marked by the sign of 'emptiness' (Sūnya). An analogous use[3] of the zero for the unknown quantity in a proportion appears in a Latin manuscript of some lectures by Gottfried Wolack in the University of Erfont in 1467 and 1468. When the Sūnya stands for the unknown, a coefficient 1 is always set under it, as for example, $\binom{y\bar{a}}{1}$ in Bhaskara's Bijaganita. Another curious feature of the Bakshālī dot-mark is that it does duty simultaneously for several

* B. M. II. Folio 22 verso and 23 recto, pp. 122, 123.

unknowns, probably because it was felt to be more general and abstract than a literal or any other symbol that might be thought of:

e. g.	·	5	*yu*	*mu*	·	*sā*	·	7†	*mu*	·	.
	1	1			1		1	1		1	

which means $\sqrt{x+5}=y$, $\sqrt{x-7}=z$.

Kaye believes that the ambiguous use of the dot-symbol is an indication of the lack of an efficient symbolism and is perplexed by the denominator unity for the dot-symbol. He fails to appreciate that the denominator unity serves to indicate that the dot-symbol is a symbol for the unknown and not the symbol for 'zero'. Here we have again another hint that the contents of the manuscript must belong to a period when the literal notation for the unknown did not come into general use in Hindu works.

The symbol for 'zero', the negative sign, and the symbol for the unknown seem to have a common ancestry and their nebulous beginnings are perhaps reflected in the Bakshālī Manuscript.

§ 6. The Rule of Regula Falsi

David Eugene Smith remarks[4] 'Awkward as this seems, the rule (explained below) was used for many centuries, a witness to the need for and value of a good symbolism'. Kaye believes in this dictum. The rule of double false to solve $ax+b=0$, gives

$$x=\frac{f_1 g_2 - f_2 g_1}{f_1 - f_2} \text{ where } ag_1+b=f_1 \text{ and } ag_2+b=f_2, \quad f_1, f_2$$

being called the two false values or failures corresponding to g_1 and g_2 respectively. In effect, I should say that this method is the same as that in analytical geometry to find the co-ordinates of the point where the line joining (g_1, f_1) and (g_2, f_2) meets the x-axis. From another point of view, the method may be deemed to involve the principle of linear interpolation. After all, does not the theory of interpolation involve the introduction of not two, but several falses hovering about the true value one is in quest of? When the truth happens to lie deep hidden, we are sometimes forced to accept the nearest falsehood as truth. Hence the method of the double false is not an awkward one but envisages an essential process of making falses themselves yield truth, a precursor of modern interpolation theory.

But the above rule ought not to be confused with the Indian Ishṭakarma which is an operation with an assumed number, used in cases where the final result arrived at after a series of manipulations is always proportional to the number originally assumed. The required number can be obtained by Trairāśika or proportion corresponding to the given final result or Driṣya. Algebraically, the Ishṭakarma is equivalent to solving the simple equation $ax=b$ and arriving at $x=\frac{bm}{am}$, where am is the value of the left-hand side when $x=m$ (Ishṭa). It is important to note that Ishtakarma is a useful device to evaluate x when a is a complicated expression, say $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})$. When a is a straightforward simple coefficient, the Hindu mathematician would straightaway utilise his Rule of Inverse Operations and get the value of x. The Ishṭakarma must be regarded therefore more as an appendix to Trairāśika (proportion) than as a degenerate case of 'Regula Falsi'.

The use of Regula Falsi in the Bakshālī Manuscript as conceived by Kaye† is not the same as the one used in the middle ages but a fanciful deduction based on a misunderstanding of an ingenious method of generalised Ishṭakarma to be presently explained. The author of Bakshālī Manuscript solves the equations

$$x_1+x_2=16,\ x_2+x_3=17,\ x_3+x_4=18,\ x_4+x_5=19,\ x_5+x_1=20$$

by assuming a tentative value (Ishṭa), say 7 for x_1 and derives from it successively the values, 9, 8, 10, 9 for x_2, x_3, x_4, x_5, and thence $x_5+x_1=16$. But from the structure of the equations, it is noticed that any decrease or increase in x_1 involves a corresponding and same decrease or increase in x_3 and x_5, so that the decrease or increase gets doubled for x_1+x_5. Hence the true value for x_1 can be obtained from the tentative value by decreasing or increasing it by *half* the total deficit or excess in the actual given value of x_1+x_5 as compared with the value based on our assumption. Thus the assumed value 7 should be increased by 2 and $x_1=9$. Thence $x_2=7$, $x_3=10$, $x_4=8$, $x_5=11$.

This method again is peculiar to Bakshālī Mathematics and I have not come across the like of it elsewhere. To trace this to the medieval Regula Falsi and thence infer an evidence that the work belongs to medieval times is a far-fetched argument.

The Hindu mathematician has never used the medieval 'Regula Falsi' to solve a simple equation of the type $ax+b=cx+d$. Kaye ought to have been aware of [5] Aryabhata's solution of this in the

† Vide B. M. I pp. 32, 33.

form $(d-b)/(a-c)$, as well as similar solutions presented in the text of Bakshālī Manuscript† and yet asserts that the Regula Falsi was used by the early Hindu mathematicians on account of lack of efficient symbolism. Regula Falsi as interpreted by Kaye, at least with reference to the Bakshālī Manuscript is regular falsehood.

§ 7. The Square-root rule and the Process of Reconciliation

Another striking feature in the mathematical contents of the Bakshālī Manuscript is the chapter on the square-root rule, where we have

$$\sqrt{a^2+\gamma} = a + \frac{\gamma}{2a} - \frac{(\gamma/2a)^2}{2\,(a+\gamma/2a)}$$

an approximation found in exactly the same form also in an Arab work of the twelfth century. This has given rise to a speculation that the present work may therefore be later than the twelfth century. But it must be noted that the rule is merely a corollary of the general square-root rule found explicitly in all Indian mathematical works, at any rate from Aryabhata onwards. The earliest evidence of a concrete numerical application of this rule even to a higher order than the one indicated by the above formula is to be found in the Sulva Sūtras, which contain the approximation $1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34}$ for $\sqrt{2}$. This formula can be easily explained by the usual Indian algorithm for finding square-roots, (the same as the one adopted even to-day):

$$
\begin{array}{r|l}
\begin{array}{r}2\\1\end{array} & 1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34} \\
\hline
2+\frac{1}{3} & \begin{array}{l}1\\ \underline{7/9}\\ 2/9\end{array} \\
2+\frac{2}{3}+\frac{1}{3\cdot4} & 2/9+\frac{1}{3\cdot4\cdot3\cdot4} \\
\cline{2-2}
2+\frac{2}{3}+\frac{1}{3\cdot2}-\frac{1}{3\cdot4\cdot34} & \quad -\frac{1}{3\cdot4\cdot3\cdot4} \\
 & \quad -\frac{1}{3\cdot4\cdot3\cdot4}+\frac{1}{3\cdot4\cdot34\cdot3\cdot4\cdot34} \\
\cline{2-2}
\end{array}
$$

and so on, ad libitum.

Further, it is important to note that the Bakshālī rule ought not to be so narrowly interpreted that it gives only the second approximation but is suggestive of a process which permits repeated application to

† B. M. III p. 173 (9 verso).

any desired order of approximation. Brahmagupta[6] mentions a similar rule in connection with the square-root of a number in sexagesimal notation.

The occasion for finding the square-root approximation is a very fanciful one. For, it is used to find the number of terms 't' of an arithmetical progression, when the sum s, the first term a and the common difference d are given. t is given by the formula:

$$t = \frac{d - 2a + \sqrt{(2a - d)^2 + 8ds}}{2d}$$

while $$s = t\left(a + \overline{t-1}.\frac{d}{2}\right)$$

Both these occur in all early Hindu mathematical works. The new feature in the present work is the following:

If an approximation to $\sqrt{(2a-d)^2 + 8ds}$ be q_1 and the corresponding value of t be t_1 and the corresponding sum s_1, the verification of the approximation consists in noting that

$$s = s_1 - \frac{e_1}{8d} \text{ where } e_1 = q_1^2 - \{(2a - d)^2 + 8ds\}$$

The proof of this result is extremely simple, but Kaye puts up a long rigmarole.

In fact, if $s_1 = t_1\left(a + \overline{t_1 - 1}.\frac{d}{2}\right)$, it follows immediately

that $$t_1 = \frac{d - 2a + \sqrt{(2a - d)^2 + 8ds_1}}{2d},$$

and hence
$$\begin{aligned} q_1^2 &= (2a - d)^2 + 8ds_1 \\ &= (2a - d)^2 + 8ds + 8d(s_1 - s); \end{aligned}$$

$$\therefore \quad e_1 = 8d\,(s_1 - s) \text{ or } s = s_1 - e_1/8d.$$

This method of verification is obviously applicable whatever be the degree of approximation. It is nothing peculiarly adapted to the approximation of the second degree. But the value of e_1 is either $\left(\frac{\gamma}{2a}\right)^2$ or $\frac{(\gamma/2a)^4}{4(a + \gamma/2a)^2}$, according as the first or second approximation is taken.

The author of the Bakshāli manuscript seems to take a peculiar pleasure in making huge calculations to verify

$$e_1 = 8d(s_1 - s)$$

in a number of examples. Mathematically, all this work is futile and meaningless, since the summation of an Arithmetical Progression to a non-integral number of terms is not practical mathematics. But such a summation seems to be very common in the days of Chaturveda, Sridhara and Mahavira, i.e. roughly the ninth and tenth centuries. Mahavira triumphantly asks "Give out the first term and the common difference respectively in relation to two series, in which $\frac{31}{150}$ is the sum, while $\frac{3}{4}$ in one case is the common difference, and $\frac{4}{5}$ the number of terms and in the other case $\frac{1}{3}$ the first term and $\frac{4}{5}$ the number of terms."[7] Very likely the Bakshali arithmetician who revels in summation of series to an irrational number of terms belongs to the same school and is perhaps contemporaneous with the writers just quoted. A typical Bakshālī problem involving square-root approximation may be given by way of illustration: 'A certain person goes 5 yojanas on the first day, and 3 yojanas more on each succeeding day. Another who travels 7 yojanas per day has a start of 5 days. When will they meet, say O! the best of mathematicians.'† The answer given in the text is $\frac{7+\sqrt{889}}{6}$ days or $6\frac{4}{29}$ days approximately, i.e. 6 days, 8 ghatikas,* 16 vighatis, 33^{iii}, $6^{iv}\frac{6}{29}$. The futility of the result transcends one's imagination. But Dr. Bibhutibhushan Datta[2] comments:

"The expression for the time in which two persons will meet contains a surd quantity. So the two persons will never meet"!! A worthy comment! The problem is quite sensible but not the above solution. The mathematician has strayed hopelessly away from common sense, carried away by his unqualified faith in the algebraic formula. The correct answer is 6 days, 7 ghatikas and 30 vighatis, quite rational and free from surds. The reader can easily verify this result.

It takes perhaps centuries to outgrow such impossible fantasies as those exhibited in the problems on square-root approximations. We are glad to find Bhaskara of the twelfth century entirely free from them. He was, perhaps, the earliest Hindu mathematician to perceive the beauty and importance of integers and integral solutions. His marvellous chakravāla method, whose import is even to this day imperfectly understood, of solving $x^2 - Ny^2 = 1$ (N a non-square positive integer) in positive integers is a remarkable instance of 'the sense' for the integer. The Bakshālī mathematician is too ancient to have cultivated this 'sense.'

† B. M. III p. 178. 6 recto.

* 1 Ghatika = $\frac{1}{60}$ day = 60 vighatsi.

§ 8. Series and Sequences

The problems on series and sequences are very original, varied and interesting. Kaye, however, holds a different opinion and gives a very confused and misleading analysis of the types of problems. In most of the problems, the sum to n terms happens to be proportional to the first term and therefore Ishtakarma can apply. The several types are indicated below in modern notation with T_n and S_n denoting respectively the n^{th} term and the sum to n terms. We discuss *eight* types according to the character of T_n.

(1) *Arithmetical Progression.*

$T_n = nT_1 \pm (a + d\cdot\overline{n-1})$†. This can be reduced to

$T_n \pm d = n(T_1 \pm d) \pm a$, or changing the notation

$V_n = nV_1 + a$, which is an arithmetic progression with common difference V_1 and first term $V_1 + a$.

(2) *Geometrical Progression.*

$$T_n = ar^{n-1} + ab\,\frac{r^{n-1}-1}{r-1} \quad ††$$

$$= a\left\{\frac{b+r-1}{r-1}\cdot r^{n-1} - \frac{b}{r-1}\right\}$$ which is derived from a G.P. by increasing every term by a constant.

$$\therefore \quad S_n = a\left\{\frac{b+r-1}{(r-1)^2}\cdot(r^n-1) - \frac{nb}{r-1}\right\}$$ which is proportional to the first term a if b, r, n be given.

Illustration: $S_n = 329$, $r = 3$, $n = 5$, $b = \frac{3}{4}$. ††

Hence $a = 2$, obtained in the text by Ishtakarma assuming $a = 1$ and setting the corresponding S_n (from which the true value of a comes out by proportion).

(3) $T_n = a\,S_{n-1}$.*

Hence $S_n = S_{n-1} + T_n = S_{n-1}(1+a) = S_{n-2}(1+a)^2 = \cdot\cdot\cdot\cdot\cdot$

$= S_1(1+a)^{n-1}$ or $T_1(1+a)^{n-1}$

(4) $T_n = n\cdot S_{n-1}$**.

This is the most interesting of all.

† B. M. II P. 122. 22 verso.

†† B. M. II P. 143, 51 verso; B. M. III p. 164.

* B. M. III P. 236 (42, verso) where $S_n = 54$. $a = \frac{1}{2}$, $n = 3$. Hence $T_1 = 24$.

** B. M. III P. 104 (23, verso) where $S_n = 300$, $n = 4$.

Hence $T_1 = 5$, $T_2 = 10$, $T_3 = 45$, $T_4 = 240$,

$$\text{Since } T_n = S_n - S_{n-1} = S_n \cdot \frac{n}{n+1},\ S_{n-1} = \frac{S_n}{n+1}.$$

$$\text{Similarly } T_{n-1} = S_{n-1} \cdot \frac{n-1}{n} = \frac{n-1}{n(n+1)} S_n \text{ and } S_{n-2} = \frac{S_n}{(n+1)n}, \cdots$$

$$T_{n-r} = \frac{(n-r)\, S_n}{(n+1)n \cdots (n-r+1)},$$

$$S_{n-r-1} = \frac{S_n}{(n+1)n \cdots (n-r+1)}\ (0 < r < n-1)$$

$$T_2 = \frac{2S_n}{(n+1)n \cdots 3} = 2T_1.$$

$$\text{Hence } T_1 = \frac{2S_n}{(n+1)!} \text{ and } S_n = \tfrac{1}{2}\, T_1 \cdot (n+1)!$$

Thus, given the law of formation of the sequence, and the sum to n terms, and the number of terms n, the series can be determined. Here again the sum is proportional to the first term and 'Ishṭa-karma' operation is applicable. This is the method adopted in the text which is indeed preferable in numerical work to the algebraic expressions that we have given.

(5) $T_n = nS_{n-1} \pm (a + \overline{n-1} \cdot d) = nS_{n-1} + U_n$ (say).

$\therefore\ S_n = (n+1)\, S_{n-1} + U_n$

$S_{n-1} = nS_{n-2} + U_{n-1}$

. . . .

$S_2 = 3S_1 + U_2 = 3T_1 + U_2$

Hence, eliminating $S_{n-1}, S_{n-2}, \cdots S_2$, we have

$$S_n = U_n + (n+1)\, U_{n-1} + (n+1) \cdot nU_{n-2} + \cdots + (n+1)\, n\, (n-1) \cdots 4U_2$$
$$+ 3 \cdot (n+1)\, n\, (n-1) \cdot \cdot 4 \cdot T_1$$
$$= \tfrac{1}{2}\, T_1 \cdot (n+1)! + U_n + (n+1)\, U_{n-1} + \cdots$$
$$+ (n+1)\, n\, (n-1) \cdots 4U_2$$

The example in the text † is somewhat like this:

A has something with $1\frac{1}{2}$, B twice as much as A with $2\frac{1}{2}$, C three times as much as A and B and $3\frac{1}{2}$ in addition, D four times as much as A, B, and C and $4\frac{1}{2}$ in addition, their total possessions are 222. What is the possession of A?

This is interpreted by Kaye symbolically as:

$A = x\ (1 + 1\frac{1}{2})$, $B = 2A + 2\frac{1}{2}x$, $C = 3A + 3B + 3\frac{1}{2}x$, $D = 4A + 4B + 4C + 4\frac{1}{2}x$, and $A + B + C + D = 222$.

† B. M. II 24 verso p. 124.

I believe this interpretation is incorrect though it may justify the solution of the text and the use of Ishṭakarma. A proper interpretation would be

$$A = x + 1\tfrac{1}{2},\ B = 2A + 2\tfrac{1}{2},\ C = 3A + 3B + 3\tfrac{1}{2},\ D = 4A + 4B + 4C + 4\tfrac{1}{2}$$

$$A + B + C + D = 222.$$

Though $x = 1$ satisfies by chance this set of equations, the set of equations is not strictly amenable to Ishṭakarma by mere proportion. A different kind of Ishṭakarma similar to the one described on p. 7 above should have been adopted here with due care. The Bakshali mathematician has not done this and therefore has erred in his method. Such an error has been repeated in three similar problems †† and we get a glimpse of the limitations of the Ganakaraja.

(6) $T =_n T_{n-1}$

Here, $T_n = n!$ if $T_1 = 1$ and there is no compendious formula for the exact summation of such terms.

(7) $T_n = nT_{n-1} \pm (a + \overline{n-1}\cdot d)$

This can be reduced to the form $U_n = nU_{n-1} \pm a$, which also does not admit of summation by any formula.

(8) $T_n = (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})a_n.$

We are reminded of Euler's identity:

$$(1 - a_1) + a_1(1 - a_2) + a_1 a_2(1 - a_3) + \cdots + a_1 a_2 \cdots a_{n-1}(1 - a_n)$$
$$= 1 - a_1 a_2 \cdots a_n.$$

Changing a_r to $1 - a_r$, we have the Indian counterpart of Euler's identity, viz.

$$a_1 + a_2(1 - a_1) + a_3(1 - a_1)(1 - a_2) + \cdots\cdots + a_n(1 - a_1)(1 - a_2)\cdots(1 - a_{n-1})$$
$$= 1 - (1 - a_1)(1 - a_2)\cdots\cdots(1 - a_n).$$

Problems involving this identity are common in Hindu mathematical works from at least the 8th century onwards. No wonder that the Arithmetical Papyrus of Akhim (9th century?) contains such problems. But Kaye would prefer to make Akhim the inventor of the above identity and hence infer that the Bakshālī Manuscript must belong to the tenth century or later.

†† B. M. II, 24 recto, 25 recto and verso, pp. 123-125.

§ 9. General Remarks

Having pointed out the famous peculiarities of the Bakshālī Manuscript in respect of its mathematical content, exposition and symbolism we will conclude with a few general remarks on the age and style of the work. Peculiarities apart, the Bakshālī text is more or less a replica of other Hindu mathematical works, such as the Ganitasara Sangraha. It contains in common with them the following:

(1) Practical and commercial problems such as the computation of fineness of gold, (2) problems on income and expenditure, (3) motion-problems, (4) profit and loss, (5) interest, (6) bills of exchange or hundika, and (7) miscellaneous problems involving literary and social references.

In addition, we may note that the way in which the solutions are given in the Bakshālī Manuscript in such a general form as to be nearly algebraic in character even though no adequate symbolism is employed, is just characteristic of all Indian works, since the day when Aryabhata set the fashion in this respect. There are, however, certain omissions, for example, expressions for the sums of squares and cubes, the indeterminate equations of the first and second degree, shadow problems, permutation and combination. These may be either apparent or real, because our manuscript is mutilated and several leaves are stuck together. In the latter case, the omission is significant and bespeaks an early date of composition, i.e. prior to the tenth century by which period these problems had become well known at least in Indian works. Among further evidences of early composition, we have already mentioned the meagre elaboration of problems dealt with in greater detail in the works of Brahmagupta Mahaviracharya and Bhaskara, the studied absence of the word-numeral notation, the use of the cross-symbol † instead of the dot ⊙ to denote a negative quantity, the use of the dot ⊙ to denote an unknown quantity, and a certain careless application of the Ishṭa-karma in as many as five problems.

The employment of the modern place-value arithmetical notation is regarded by Kaye as indicating a late period, as also what he believes to be non-Indian material, viz. the negative sign resembling the Diophantine symbol, the use of the Regula-falsi, the square-root approximation, the use of the Sexagesimal notation for example in the conversion of days into ghatikas, vighatis, and so forth, and the use of the terms dinara, dramma, satera and so forth. How much the non-Indian material in Bakshālī Manuscript is really indigenous has been sufficiently brought out in the preceding pages.

However late we may wish to place the manuscript, we cannot well go beyond the tenth century. We therefore come near the times of Sridhara, Mahavira, and Chaturveda with whose works the Bakshālī work has many points in common. Evidences of language, script and such special terms as Hundika do not contradict this view. Kaye's insistence of the twelfth century as the probable date of composition is wholly untenable.

The systematic presentation of the working steps and methods of verification, along with the use of a certain kind of syncopated notation are of great pedagogic interest, quite in keeping with the fact that the work was written for the express benefit of young boys†. The simple and diffuse style adopted suits this aim.

The Bakshālī Manuscript may lack the subtlety of the Aryabhatiya; it may not possess the profundity of the Brahmasphuta Siddhanta; it may be a poor literary specimen by the side of the rich poetry of Ganitasarasangraha; but no one can deny its great value for a practical teacher. If we recollect that Bakshālī was at a distance of only 70 miles from Taxila, one of the renowned University centres in Ancient India, we have here perhaps a glimpse of the lecture-notes of a University professor of bygone days. To do the Ganakaraja justice, we may add that flashes of genius gleam forth here and there in the method of reconciliation of square-root approximations, in dreaming of series and sequences governed by elegant and sometimes complicated laws, in the solution of systems of linear equations by suitable change of variables and assumption of tentative solutions, and lastly in the construction of problems of humorous and mythological interest.

References

1. *The Bakshālī Manuscript* by G. R. Kaye, Archaeological Survey of India, New Imperial Series, Vol. XLIII, Parts I and II, (1927) and Part III (1933).

2. The Bakshālī Mathematics by Bibhutibhusan Datta, *Bulletin of the Cal. Math. Soc.* XXI, 1929.

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† B. M. II p. 142, 50r. The Colophon reads:
.... Vasiṣhta putraha.
Sikasyartheputra pautra upayogam bhavatu likhi.
tam chchhajaka putra gaṇaka rāja brahmaṇena . . .

3. *The Hindu Arabic Numerals*, Smith and Karpinski.

4. *History of Mathematics*, David Eugene Smith, Vol. II.

5. *Aryabhatiyam*, Kern's Edition, Leiden, 1874.

6. *Brahmasiddhanta* of Brahmagupta, Edited by Sudhakara Dvivedi, Benares, 1902.

7. *Ganitasara Sangraha* by Mahaviracharya, with English Translation and Notes, Madras 1912.